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RIEMANNIAN MANIFOLDS ADMITTING A CONFORMAL TRANSFORMATION GROUP

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1. Introduction

The purpose of the present paper is to generalize some of the known results on Riemannian manifolds with constant scalar curvature admitting a group of nonisometric conformal transformations.

Let M be a connected Riemannian manifold of dimension n, and g_{ji}, ∇_i, $K_{kji}{}^h$, $K_{ji} = K_{tji}{}^t$ and $K = K_{ji}g^{ji}$, respectively, the positive definite fundamental metric tensor, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $1, \cdots, n$.

If we put

$$G_{ji} = K_{ji} - \frac{K}{n} g_{ji} , \tag{1.1}$$

$$Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{n(n-1)} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) , \tag{1.2}$$

we have

$$Z_{tji}{}^t = G_{ji} , \quad G_{ji}g^{ji} = 0 . \tag{1.3}$$

When M admits an infinitesimal transformation v^h, we denote by $\mathscr{L}$ the operator of Lie derivation with respect to v^h. Thus, if M admits an infinitesimal conformal transformation v^h, we have

$$\mathscr{L} g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \quad \mathscr{L} g^{ih} = -2\rho g^{ih} \tag{1.4}$$

for a certain scalar field ρ. We denote the gradient of ρ by $\rho_i = \nabla_i \rho$.

For an infinitesimal conformal transformation v^h in M, we have [5]

$$\mathscr{L} K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} , \tag{1.5}$$

$$\mathscr{L} K_{ji} = -(n-2) \nabla_j \rho_i - \Delta \rho g_{ji} , \tag{1.6}$$

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$$\mathscr{L}K = -2(n-1)\Delta\rho - 2\rho K\,, \tag{1.7}$$

where

$$\Delta\rho = g^{ji}\nabla_j\nabla_i\rho\,. \tag{1.8}$$

Thus, in M with $K=$ const. we have

$$\Delta\rho = -\frac{K}{n-1}\rho. \tag{1.9}$$

We also have

$$\mathscr{L}G_{ji} = -(n-2)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\,, \tag{1.10}$$

$$\begin{aligned}\mathscr{L}Z_{kji}{}^h = &-\delta_k^h\nabla_j\rho_i + \delta_j^h\nabla_k\rho_i - \nabla_k\rho^h g_{ji} + \nabla_j\rho^h g_{ki}\\ &+\frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki})\,.\end{aligned} \tag{1.11}$$

Thus, in M with $K=$ const. we have

$$\mathscr{L}G_{ji} = -(n-2)\left[\nabla_j\rho_i + \frac{K}{n(n-1)}\rho g_{ji}\right]\,. \tag{1.12}$$

We denote by $C_0(M)$ the largest connected group of conformal transformations of M and by $I_0(M)$ that of isometries of M.

We first state here known results on Riemannian manifolds with $K=$ const. admitting a conformal transformation group, and then try to generalize them.

Theorem A (*Lichnerowicz* [3]). *If M is a compact Riemannian manifold of dimension $n>2$, $K=$ const., $K_{ji}K^{ji}=$ const., and $C_0(M)\neq I_0(M)$, then M is isometric to a sphere.*

Theorem B (*Lichnerowicz* [3], *Yano & Obata* [7]). *If a compact Riemannian manifold M of dimension $n\geq 2$ with $K=$ const. admits an infinitesimal nonisometric conformal transformation $v^h : \mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho\neq$ const., and if one of the following conditions is satisfied, then M is isometric to a sphere:*

(1) *The vector field v^h is a gradient of a scalar.*

(2) *$K_i{}^h\rho^i = k\rho^h$, k being a constant.*

(3) *$\mathscr{L}K_{ji} = \alpha g_{ji}$, α being a scalar field.*

Theorem C (*Hsiung* [1]). *If M is compact and of dimension $n>2$, $K=$ const., $K_{kjih}K^{kjih}=$ const., and $C_0(M)\neq I_0(M)$, then M is isometric to a sphere.*

Theorem D (*Yano* [6]). *If M is compact orientable and of dimension $n>2$ with $K=$ const., and admits an infinitesimal nonisometric conformal* *transformation* $v^h: \mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho \neq$ *const., such that* $\int_M G_{ji}\rho^j\rho^i dV$ *is non-negative,* dV *being the volume element of* M, *then* M *is isometric to a sphere.*

Theorem E (*Yano* [6]). *If* M *is a compact and of dimension* $n > 2$ *with* $K =$ *const., and admits an infinitesimal nonisometric conformal transformation* $v^h: \mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho \neq$ *const., such that* $\mathscr{L}(G_{ji}G^{ji}) =$ *const. or* $\mathscr{L}(Z_{kjih}Z^{kjih})$ $=$ *const., then* M *is isometric to a sphere.*

Theorem F (*Hsiung* [2]). *Suppose that a compact Riemannian manifold* M *of dimension* $n > 2$ *with* $K =$ *const. admits an infinitesimal nonhomothetic conformal transformation* v^h. *If*

$$a^2\mathscr{L}(Z_{kjih}Z^{kjih}) + (2a + nb)b\mathscr{L}(G_{ji}G^{ji}) = \text{const.}\,, \tag{1.13}$$

where a *and* b *are constants such that*

$$c \equiv 4a^2 + 2(n-2)ab + n(n-2)b^2 > 0\,, \tag{1.14}$$

then M *is isometric to a sphere.*

To prove and generalize these theorems, we need the following

Theorem G (*Obata* [4]). *If a complete Riemannian manifold of dimension* $n \geq 2$ *admits a nonconstant function* ρ *such that*

$$\nabla_j \nabla_i \rho = -c^2 \rho g_{ji}\,, \tag{1.15}$$

where c *is a positive constant, then* M *is isometric to a sphere of radius* $1/c$ *in* $(n+1)$*-dimensional Euclidean space.*

We also need following integral formulas proved in [6].

If a compact orientable Riemannian manifold M of dimension $n > 2$ with $K =$ const. admits an infinitesimal nonhomothetic conformal transformation $v^h: \mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho \neq$ const., then we have

$$\int_M G_{ji}\rho^j\rho^i dV = \frac{1}{n-2}\int_M \left[2\rho^2 G_{ji}G^{ji} + \frac{1}{2}\rho\mathscr{L}(G_{ji}G^{ji})\right]dV\,, \tag{1.16}$$

$$\int_M G_{ji}\rho^j\rho^i dV = \int_M \left[\frac{1}{2}\rho^2 Z_{kjih}Z^{kjih} + \frac{1}{8}\rho\mathscr{L}(Z_{kjih}Z^{kjih})\right]dV\,. \tag{1.17}$$

2. Generalization of Theorem B, (2), (3)

Theorem 2.1. *If a compact orientable* M *of dimension* $n > 2$ *with* K $=$ *const. admits an infinitesimal nonhomothetic conformal transformation* $v^h: \mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho \neq$ *const., such that*

$$\mathscr{L}(G^{ji}\mathscr{L}G_{ji}) \leq 0\,, \tag{2.1}$$

then M is isometric to a sphere.

We need the following

Lemma 2.1. *If a compact orientable M admits an infinitesimal conformal tsansformation* $v^h : \mathscr{L}g_{ji} = 2\rho g_{ji}$, *then we have*

$$\int_M \rho F dV = -\frac{1}{n}\int_M \mathscr{L}F dV \tag{2.2}$$

for any function F.

Proof. Since $\rho = \dfrac{1}{n}\nabla_s v^s$, we have, by Green's theorem,

$$\begin{aligned}\int_M \rho F dV &= \frac{1}{n}\int_M (\nabla_s v^s) F dV \\ &= -\frac{1}{n}\int_M v^s \nabla_s F dV \\ &= -\frac{1}{n}\int_M \mathscr{L}F dV\,.\end{aligned}$$

Proof of the Theorem. Substituting

$$\mathscr{L}(G_{ji}G^{ji}) = 2G^{ji}\mathscr{L}G_{ji} - 4\rho G^{ji}G_{ji} \tag{2.3}$$

into integral formula (1.16), we find

$$\int_M G_{ji}\rho^j\rho^i dV = \frac{1}{n-2}\int_M \rho G^{ji}\mathscr{L}G_{ji} dV\,. \tag{2.4}$$

Consequently, by Lemma 2.1 and the assumption of the theorem, we have

$$\int_M G_{ji}\rho^j\rho^i dV = -\frac{1}{n(n-2)}\int_M \mathscr{L}(G^{ji}\mathscr{L}G_{ji}) dV \geq 0\,.$$

Thus M is isometric to a sphere by Theorem D.

Remark 2.1. Since

$$Z^{kji}{}_h\mathscr{L}Z_{kji}{}^h = \frac{4}{n-2}G^{ji}\mathscr{L}G_{ji}\,, \tag{2.5}$$

the condition (2.1) of the theorem can be replaced by

$$\mathscr{L}(Z^{kji}{}_h\mathscr{L}Z_{kji}{}^h) \leq 0\,. \tag{2.6}$$

Remark 2.2. As the proof of the theorem shows, condition (2.1) can be replaced by

$$\mathscr{L}(G^{ji}\mathscr{L}G_{ji}) = \lambda\,, \qquad \int_M \lambda dV \leq 0\,. \tag{2.7}$$

The same remark applies to Theorems 4.1, 4.3, 5.1, 6.1, 6.2 and 6.4.

Remark 2.3. Theorem 2.1 generalizes Theorem B, (2). In fact, using $K_i{}^h\rho^i = k\rho^h$, $\nabla_j K^{ji} = 0$, $\nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}$ and $\nabla_i v^i = n\rho$, we have

$$\begin{aligned}\nabla_j(K^{ji}\rho v_i) &= K^{ji}\rho_j v_i + K^{ji}\rho\nabla_j v_i\\ &= k\rho_i v^i + \frac{1}{2}K^{ji}\rho(\nabla_j v_i + \nabla_i v_j)\\ &= k\nabla_i(\rho v^i) - k\rho\nabla_i v^i + K\rho^2\\ &= k\nabla_i(\rho v^i) - nk\rho^2 + K\rho^2\,,\end{aligned}$$

from which, by integration,

$$\int_M (K - nk)\rho^2 dV = 0\,,$$

and consequently $k = K/n$.

Thus, from $K_i{}^h\rho^i = k\rho^h$ we have

$$\left(K^{ji} - \frac{K}{n}g^{ji}\right)\rho_i = 0\,,$$

$$\left(K^{ji} - \frac{K}{n}g^{ji}\right)\nabla_j\rho_i = 0\,,$$

and consequently, by virtue of (1.10),

$$G^{ji}(\mathscr{L}G_{ji}) = 0\,.$$

Remark 2.4. Theorem 2.1 generalizes Theorem B, (3). In fact, from (1.6) and $\mathscr{L}K_{ji} = \alpha g_{ji}$ we find

$$-(n-2)\nabla_j\rho_i - \Delta\rho g_{ji} = \alpha g_{ji}\,,$$

from which

$$\alpha = -2(n-1)\Delta\rho/n\,,$$

and consequently

$$-(n-2)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right) = 0\,,$$

that is, $\mathscr{L}G_{ji} = 0$.

Remark 2.5. If $G^{ji}\mathscr{L}G_{ji} = \text{const.}$, then (2.1) is automatically satisfied, but under our assumption the constant must be zero. In fact, making use of (1.3) and $\nabla_j G^{ji} = 0$, from (1.10) we have

$$\begin{aligned} G^{ji}\mathscr{L}G_{ji} &= -(n-2)G^{ij}\nabla_j \rho_i \\ &= -(n-2)\nabla_j(G^{ji}\rho_i) , \end{aligned}$$

and consequently by integration over M we find

$$\int_M G^{ji}\mathscr{L}G_{ji} dV = 0 .$$

Thus, if $G^{ji}\mathscr{L}G_{ji} = \text{const.}$ the constant must be zero.

3. Decomposition of a conformal Killing vector

Theorem 3.1. *If a compact orientable M of dimension $n \geq 2$ with $K = \text{const.}$ admits a conformal Killing vector field*

$$v^h = p^h + q^h , \tag{3.1}$$

where p^h is a Killing vector field and $q^h = \nabla^h q$, $q \neq \text{const.}$ is a gradient conformal Killing vector field, then M is isometric to a sphere.

Conversely, if a sphere of dimension $n \geq 2$ admits a conformal Killing vector field v^h, then v^h is decomposed into the form (3.1) *where p^h is a Killing vector field and q^h a gradient conformal Killing vector field.*

Proof. Suppose that a compact orientable M with $K = \text{const.}$ admits a conformal Killing vector v^h. Then we have

$$\mathscr{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} , \tag{3.2}$$

and

$$\Delta\rho = -\frac{K}{n-1}\rho . \tag{3.3}$$

We note here that K is a positive constant [6]. If v^h is the sum of a Killing vector p^h and a gradient conformal Killing vector $q^h = \nabla^h q$, substituting (3.1) into (3.2), we find

$$\nabla_j \nabla_i q = \rho g_{ji} , \tag{3.4}$$

from which

$$\Delta q = n\rho . \tag{3.5}$$

From (3.3) and (3.5), we find

$$\Delta\left(\rho + \frac{K}{n(n-1)}q\right) = 0 ,$$

from which, by Bochner's lemma,

$$\rho + \frac{K}{n(n-1)}q = \text{constant} . \tag{3.6}$$

Substituting (3.6) into (3.4), we find

$$\nabla_j \nabla_i (q + c) = -\frac{K}{n(n-1)}(q + c)g_{ji} ,$$

where c is a constant. Thus, q being not a constant, M is isometric to a sphere.

Conversely, suppose that M, isometric to a sphere, admits a conformal Killing vector v^h. It is known that v^h can be decomposed into

$$v^h = p^h + q^h ,$$

where

$$\nabla_i p^i = 0 , \qquad q^h = \nabla^h q . \tag{3.7}$$

From

$$\mathscr{L} g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} ,$$

we have

$$T_{ji} = \nabla_j p_i + \nabla_i p_j + 2\nabla_j \nabla_i q - 2\rho g_{ji} = 0 . \tag{3.8}$$

Forming $T_{ji}T^{ji}$, we find

$$\begin{aligned} T_{ji}T^{ji} &= (\nabla_j p_i + \nabla_i p_j)(\nabla^j p^i + \nabla^i p^j) \\ &\quad + 4\left(\nabla_j \nabla_i q - \frac{1}{n}\Delta q g_{ji}\right)\left(\nabla^j \nabla^i q - \frac{1}{n}\Delta q g^{ji}\right) \\ &\quad + 8(\nabla^j \nabla^i q)(\nabla_j p_i) = 0 . \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \int_M (\nabla^j \nabla^i q)(\nabla_j p_i) dV &= \int_M (\nabla^i \nabla^j q)(\nabla_j p_i) dV \\ &= -\int_M (\nabla^j q)(\nabla^i \nabla_j p_i) dV \\ &= -\int_M K_{ji}(\nabla^j q) p^i dV \end{aligned}$$

because of

$$\nabla_i \nabla_j p^i - \nabla_j \nabla_i p^i = K_{ijt}{}^i p^t ,$$

or

$$\nabla^i \nabla_j p_i = K_{jt} p^t .$$

Taking account of $K_{ji} = \dfrac{K}{n} g_{ji}$ we then have

$$\int_M (\nabla^j \nabla^i q)(\nabla_j p_i) dV = -\frac{K}{n} \int_M (\nabla_i q) p^i dV$$
$$= \frac{K}{n} \int_M q(\nabla_i p^i) dV$$
$$= 0 .$$

Thus from (3.9), by integration we find

$$\int_M \Big[(\nabla_j p_i + \nabla_i p_j)(\nabla^j p^i + \nabla^i p^j) + 4\Big(\nabla_j \nabla_i q - \frac{1}{n} \Delta q g_{ji}\Big)\Big(\nabla^j \nabla^i q - \frac{1}{n} \Delta q g^{ji}\Big)\Big] dV = 0 ,$$

from which

$$\nabla_j p_i + \nabla_i p_j = 0 , \tag{3.10}$$

$$\nabla_j \nabla_i q = \frac{1}{n} \Delta q g_{ji} , \tag{3.11}$$

showing that p^h is a Killing vector field and q^h a gradient conformal Killing vector field.

Remark 3.1. Theorem 3.1 generalizes Theorem B, (1).

Remark 3.2. We can see in the following way the fact that a sphere admits a gradient conformal Killing vector field. Let

$$X^A = X^A(x^h) , \qquad \sum X^A X^A = r^2 \tag{3.12}$$

be the equations of n-dimensional sphere of radius r in an $(n+1)$-dimensional Euclidean space, where $A = 1, \cdots, n+1$.

The equations of Gauss and those of Weingarten of the shpere are, respectively,

$$\nabla_j B_i{}^A = \frac{1}{r} g_{ji} N^A , \tag{3.13}$$

and

$$\nabla_j N^A = -\frac{1}{r} B_j{}^A , \tag{3.14}$$

where $B_i{}^A = \nabla_i X^A$ and N^A are components of the unit normal to the sphere.

Considering a parallel vector field $B_i{}^A u^i + \alpha N^A$ in the Euclidean space along the sphere, we have

$$\nabla_j(B_i{}^A u^i + \alpha N^A) = 0 ,$$

from which

$$\frac{1}{r} u_j N^A + B_i{}^A \nabla_j u^i + (\nabla_j \alpha) N^A - \frac{\alpha}{r} B_j{}^A = 0 ,$$

and consequently

$$\nabla_j u^i = \frac{\alpha}{r} \delta_j^i , \qquad \Delta_j \alpha = -\frac{1}{r} u_j ,$$

thus giving

$$\nabla_j \nabla_i \alpha = -\frac{1}{r} \nabla_j u_i ,$$

that is,

$$\nabla_j \nabla_i \alpha = -\frac{1}{r^2} \alpha g_{ji} .$$

4. Generalizations of Theorem E

We introduce here the notations:

$$f = G_{ji} G^{ji} , \qquad g = Z_{kjih} Z^{kjih} . \tag{4.1}$$

Theorem 4.1. *If a compact orientable M of dimension $n > 2$ with $K =$ const. admits an infinitesimal nonhomothetic conformal transformation v^h such that*

$$\mathscr{L}\left\{\sum_{k=0}^{l} \alpha_k \left(-\frac{n-1}{K}\right)^k \Delta^k(\mathscr{L} f) + \sum_{k=0}^{m} \beta_k \left(-\frac{n-1}{K}\right)^k \Delta^k(\mathscr{L} g)\right\} \leq 0 , \tag{4.2}$$

l and m being nonnegative integers, and α_k and β_k constants such that the sums $\sum_{k=0}^{l} \alpha_k$ and $\sum_{k=0}^{m} \beta_k$ are nonnegative and not both zero, then M is isometric to a sphere.

We need the following

Lemma 4.1. *If a compact orientable M with $K = const.$ admits an infinitesimal conformal transformation $v^h : \mathscr{L}g_{ji} = 2\rho g_{ji}$, then we have*

$$\begin{aligned}\int_M \rho F dV &= \int_M \left(-\frac{n-1}{K}\right)\rho\Delta F dV \\ &= \int_M \left(-\frac{n-1}{K}\right)^2 \rho\Delta^2 F dV \qquad (4.3)\\ &\cdots\cdots \\ &= \int_M \left(-\frac{n-1}{K}\right)^l \rho\Delta^l F dV\end{aligned}$$

for any function F and any nonnegative integer l.

Proof. Remembering

$$\Delta\rho = -\frac{K}{n-1}\rho \qquad (K > 0),$$

that is,

$$\rho = -\frac{n-1}{K}\Delta\rho,$$

we have, for any scalar field F,

$$\int_M \rho F dV = \int_M \left(-\frac{n-1}{K}\right)(\Delta\rho) F dV,$$

that is,

$$\int_M \rho F dV = \int_M \left(-\frac{n-1}{K}\right)\rho\Delta F dV.$$

Repeating the same process, we hence obtain (4.3).

Proof of the theorem. We have, from (1.16) and Lemma 4.1,

$$\begin{aligned}\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L} f dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)\rho\Delta\mathscr{L} f dV \\ &\cdots\cdots \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)^l \rho\Delta^l\mathscr{L} f dV.\end{aligned}$$

We also have, from (1.17) and Lemma 4.1,

$$\begin{aligned}2\int_M G_{ji}\rho^j\rho^i dV &= \int_M \rho^2 g dV + \frac{1}{4}\int_M \rho\mathscr{L} g dV\\ &= \int_M \rho^2 g dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)\rho\Delta\mathscr{L} g dV\\ &\qquad \cdots\cdots\\ &= \int_M \rho^2 g dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)^m \rho\Delta^m\mathscr{L} g dV\,.\end{aligned}$$

From these equations, we have

$$\begin{aligned}&\left\{\frac{n-2}{2}(\alpha_0+\alpha_1+\cdots+\alpha_l)+2(\beta_0+\beta_1+\cdots+\beta_m)\right\}\int_M G_{ji}\rho^j\rho^i dV\\ &= \int_M \rho^2\{(\alpha_0+\alpha_1+\cdots+\alpha_l)f+(\beta_0+\beta_1+\cdots+\beta_m)g\}dV\\ &\quad+\frac{1}{4}\int_M \rho\left\{\alpha_0\mathscr{L}f+\alpha_1\left(-\frac{n-1}{K}\right)\Delta\mathscr{L}f+\cdots+\alpha_l\left(-\frac{n-1}{K}\right)^l\Delta^l\mathscr{L}f\right.\\ &\qquad\qquad+\beta_0\mathscr{L}g+\beta_1\left(-\frac{n-1}{K}\right)\Delta\mathscr{L}g+\cdots\\ &\qquad\qquad\left.+\beta_m\left(-\frac{n-1}{K}\right)^m\Delta^m\mathscr{L}g\right\}dV\,,\end{aligned}$$

and consequently, by Lemma 2.1,

$$\begin{aligned}&\left\{\frac{n-2}{2}(\alpha_0+\alpha_1+\cdots+\alpha_l)+2(\beta_0+\beta_1+\cdots\beta_m)\right\}\int_M G_{ji}\rho^j\rho^i dV\\ &= \int_M \rho^2\{(\alpha_0+\alpha_1+\cdots+\alpha_l)f+(\beta_0+\beta_1+\cdots+\beta_m)g\}dV\\ &\quad-\frac{1}{4n}\int_M \mathscr{L}\left\{\alpha_0\mathscr{L}f+\alpha_1\left(-\frac{n-1}{K}\right)\Delta\mathscr{L}f+\cdots\right.\\ &\qquad\qquad+\alpha_l\left(-\frac{n-1}{K}\right)^l\Delta^l\mathscr{L}f+\beta_0\mathscr{L}g\\ &\qquad\qquad+\beta_1\left(-\frac{n-1}{K}\right)\Delta\mathscr{L}g+\cdots\\ &\qquad\qquad\left.+\beta_m\left(-\frac{n-1}{K}\right)^m\Delta^m\mathscr{L}g\right\}dV\,.\end{aligned}$$

Thus, if the conditions of the theorem are satisfied, we have

$$\int_M G_{ji}\rho^j\rho^i dV \geq 0 ,$$

and consequently, by Theorem D, M is isometric to a sphere.

Theorem 4.2. *Suppose that a compact orientable M of dimension $n > 2$ with $K = const.$ satisfies*

$$\alpha_0 f - \alpha_1 \Delta f + \beta_0 g - \beta_1 \Delta g = const. , \tag{4.4}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are nonnegative constants not all zero such that, if $n > 6$,

$$\frac{8K}{n-1}\alpha_1 \geq (n-6)\alpha_0 \geq 0 , \quad \frac{8K}{n-1}\beta_1 \geq (n-6)\beta_0 \geq 0 . \tag{4.5}$$

If M admits an infinitesimal nonhomothetic conformal transformation v^h: $\mathscr{L}g_{ji} = 2\rho g_{ji}$, $\rho \neq constant$, then M is isometric to a sphere.

To prove the theorem, we need the following

Lemma 4.2. *For a conformal Killing vector v^h in M, that is, for a vector field v^h satisfying*

$$\mathscr{L}g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} ,$$

we heve

$$\Delta(\mathscr{L}F) = \mathscr{L}(\Delta F) + 2\rho\Delta F - (n-2)\rho^i\nabla_i F \tag{4.6}$$

for any scalar field F.

Proof. Since v^h is a conformal Killing vector field, we have

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + (n-2)\rho^h = 0 , \tag{4.7}$$

(see [6] for example). We also have, for an arbitrary scalar field F,

$$g^{ji}\nabla_j\nabla_i\nabla_h F - K_h{}^i\nabla_i F = \nabla_h(\Delta F) . \tag{4.8}$$

Thus we have

$$\begin{aligned}\Delta(\mathscr{L}F) &= g^{ji}\nabla_j\nabla_i(v^h\nabla_h F)\\ &= (g^{ji}\nabla_j\nabla_i v^h)\nabla_h F + (\nabla^j v^i + \nabla^i v^j)\nabla_j\nabla_i F\\ &\quad + v^h g^{ji}\nabla_j\nabla_i\nabla_h F ,\end{aligned}$$

and consequently, by using (4.7) and (4.8),

$$\begin{aligned}\Delta(\mathscr{L}F) = &-K_{ji}v^i\nabla^j F - (n-2)\rho^h\nabla_h F\\ &+ 2\rho\Delta F + K_{ji}v^j\nabla^i F + v^h\nabla_h(\Delta F) ,\end{aligned}$$

that is,

$$\Delta(\mathscr{L}F) = \mathscr{L}\Delta F + 2\rho\Delta F - (n-2)\rho^h \nabla_h F .$$

Lemma 4.3. *For any scalar field F and a scalar field ρ satisfying $\Delta\rho = k\rho$, k being a constant, in a compact orientable M we have*

$$\int_M \rho\rho^h \nabla_h F dV = -\frac{1}{2}\int_M \rho^2(\Delta F)dV , \tag{4.9}$$

$$\int_M \rho^2(\Delta F)dV = 2k\int_M \rho^2 F dV + 2\int_M \rho_i\rho^i F dV . \tag{4.10}$$

Proof. Integral formula (4.9) follows from

$$\nabla^h(\rho^2\nabla_h F) = 2\rho\rho^h\nabla_h F + \rho^2\Delta F$$

by integration. On the other hand, we have

$$\begin{aligned}\int_M \rho^2(\Delta F)dF &= \int_M (\Delta\rho^2)FdV \\ &= 2\int_M (\rho\Delta\rho + \rho_i\rho^i)FdV \\ &= 2k\int_M \rho^2 FdV + 2\int_M \rho_i\rho^i FdV ,\end{aligned}$$

which proves (4.10).

Proof of the theorem. From (1.16), (4.3), (4.6) and (4.9), we find

$$\begin{aligned}&\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L}f dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)\rho\Delta\mathscr{L}f dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \left(-\frac{n-1}{K}\right)\rho[\mathscr{L}\Delta f + 2\rho\Delta f - (n-2)\rho^i\nabla_i f]dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L}\left(-\frac{n-1}{K}\Delta f\right)dV \\ &\quad + \frac{1}{4}\int_M \left[-\frac{2(n-1)}{K}\rho^2\Delta f - \frac{(n-1)(n-2)}{2K}\rho^2\Delta f\right]dV \\ &= \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L}\left(-\frac{n-1}{K}\Delta f\right)dV - \frac{(n-1)(n+2)}{8K}\int_M \rho^2\Delta f dV .\end{aligned}$$

Thus, we have

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L} f dV ,$$

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L}\left(-\frac{n-1}{K}\Delta f\right) dV$$
$$- \frac{(n-1)(n+2)}{8K}\int_M \rho^2 \Delta f dV .$$

Similarly, we find

$$2\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 g dV + \frac{1}{4}\int_M \rho\mathscr{L} g dV ,$$

$$2\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 g dV + \frac{1}{4}\int_M \rho\mathscr{L}\left(-\frac{n-1}{K}\Delta g\right) dV$$
$$- \frac{(n-1)(n+2)}{8K}\int_M \rho^2 \Delta g dV .$$

From the above four equations, we obtain

$$\begin{aligned}
&\left\{\frac{n-2}{2}(a+a') + 2(b+b')\right\}\int_M G_{ji}\rho^j\rho^i dV \\
&= \int_M \rho^2[(a+a')f + (b+b')g] dV \\
&\quad + \frac{1}{4}\int_M \rho\left[\mathscr{L}\left(af - \frac{n-1}{K}a'\Delta f + bg - \frac{n-1}{K}b'\Delta g\right)\right] dV \\
&\quad - \frac{(n-1)(n+2)}{8K}\int_M \rho^2(a'\Delta f + b'\Delta g) dV ,
\end{aligned} \tag{4.11}$$

a, a', b, b' being nonnegative constants. Now we choose a, a', b, b' in such a way that we have

$$\alpha_0 = a , \quad \alpha_1 = \frac{n-1}{K}a' , \quad \beta_0 = b , \quad \beta_1 = \frac{n-1}{K}b' . \tag{4.12}$$

Then we have, from (4.4),

$$af - \frac{n-1}{K}a'\Delta f + bg - \frac{n-1}{K}b'\Delta g = c \text{ (const.)} \tag{4.13}$$

and

$$a'\Delta f + b'\Delta g = \frac{K}{n-1}(af + bg) - \frac{Kc}{n-1},$$

and consequently, from (4.11),

$$\left\{\frac{n-2}{2}(a + a') + 2(b + b')\right\}\int_M G_{ji}\rho^j\rho^i dV$$
$$= \int_M \rho^2\Big[(a + a')f + (b + b')g - \frac{(n-1)(n+2)}{8K}\left\{\frac{K}{n-1}(af + bg) - \frac{Kc}{n-1}\right\}\Big]dV,$$

that is,

$$\left\{\frac{n-2}{2}(a + a') + 2(b + b')\right\}\int_M G_{ji}\rho^j\rho^i dV$$
$$= \frac{1}{8}\int_M \rho^2\Big[\{8a' - (n-6)a\}f + \{8b' - (n-6)b\}g + (n+2)c\Big]dV. \tag{4.14}$$

Now, constants

$$8a' - (n-6)a, \qquad 8b' - (n-6)b$$

are both nonnegative for $n \leq 6$. Since

$$8a' - (n-6)a = \frac{8K}{n-1}\alpha_1 - (n-6)\alpha_0,$$

$$8b' - (n-6)b = \frac{8K}{n-1}\beta_1 - (n-6)\beta_0,$$

they are nonnegative also for $n \geq 6$ by virtue of the assumption.

Moreover, we have, from (4.13),

$$a\int_M f dV + b\int_M g dV = c\int dV,$$

and consequently c is nonnegative.

Thus we have, from (4.14),

$$\int_M G_{ji}\rho^j\rho^i dV \geq 0,$$

and consequently, by Theorem D, M is isometric to a sphere.

Theorem 4.3. *If a compact orientable M of dimension $n > 2$ with $K = const.$ admits an infinitesimal nonhomothetic conformal transformation v^h such that*

$$\mathscr{L}\mathscr{L}(\alpha_0 f + \alpha_1 \Delta f + \beta_0 g + \beta_1 \Delta g) \leq 0 , \tag{4.15}$$

$\alpha_0, \alpha_1, \beta_0, \beta_1$ being constants not all zero such that

$$\frac{4(n-1)}{K}\alpha_0 \geq (n+6)\alpha_1 \geq 0 , \quad \frac{4(n-1)}{K}\beta_0 \geq (n+6)\beta_1 \geq 0 , \tag{4.16}$$

then M is isometric to a sphere.

To prove this theorem, we need the following

Lemma 4.4. *If a compact orientable M of dimension $n > 2$ with $K = const.$ admits an infinitesimal nonhomothetic conformal transformation v^h, then*

$$\begin{aligned} \frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = {} & \frac{n+6}{4}\int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta f dV \\ & - \frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i f dV , \end{aligned} \tag{4.17}$$

$$\begin{aligned} 2\int_M G_{ji}\rho^j\rho^i dV = {} & \frac{n+6}{4}\int_M \rho^2 g dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta g dV \\ & - \frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i g dV . \end{aligned} \tag{4.18}$$

Proof. From (1.16), we have

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L} f dV .$$

Substituting $\rho = -\dfrac{n-1}{K}\Delta\rho$ into the last term of the second member of this equation, we find

$$\begin{aligned} \frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV & = \int_M \rho^2 f dV - \frac{n-1}{4K}\int_M (\Delta\rho)\mathscr{L} f dV \\ & = \int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\Delta(\mathscr{L} f) dV , \end{aligned}$$

and consequently, by (4.6),

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\{\mathscr{L}\Delta f + 2\rho\Delta f - (n-2)\rho^i\nabla_i f\} dV .$$

Thus by (4.9) and (4.10) with $k = -\dfrac{K}{n-1}$, we find

$$\begin{aligned}\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV &= \int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \left(\rho\mathscr{L}\Delta f + \frac{n+2}{2}\rho^2\Delta f\right) dV \\ &= \int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{8K}\int_M \left(-\frac{2K}{n-1}\rho^2 f + 2\rho_i\rho^i f\right) dV \\ &= \frac{n+6}{4}\int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta f dV \\ &\quad - \frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i f dV .\end{aligned}$$

We can similarly prove (4.18).

Proof of the theorem. We first write down (1.16), (4.17), (1.17) and (4.18):

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 f dV + \frac{1}{4}\int_M \rho\mathscr{L} f dV ,$$

$$\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dV = \frac{n+6}{4}\int_M \rho^2 f dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta f dV - \frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i f dV ,$$

$$2\int_M G_{ji}\rho^j\rho^i dV = \int_M \rho^2 g dV + \frac{1}{4}\int_M \rho\mathscr{L} g dV ,$$

$$2\int_M G_{ji}\rho^j\rho^i dV = \frac{n+6}{4}\int_M \rho^2 g dV - \frac{n-1}{4K}\int_M \rho\mathscr{L}\Delta g dV - \frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i g dV ,$$

from which we obtain

$$\left\{\frac{n-2}{2}(a-a')+2(b-b')\right\}\int_M G_{ji}\rho^j\rho^i dV$$
$$=\frac{1}{4}\{4a-(n+6)a'\}\int_M \rho^2 f dV+\frac{1}{4}\{4b-(n+6)b'\}\int_M \rho^2 g dV$$
$$+\frac{1}{4}\int_M \rho\mathscr{L}\left(af+\frac{n-1}{K}a'\Delta f+bg+\frac{n-1}{K}b'\Delta g\right)dV$$
$$+\frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i(a'f+b'g)dV\,,$$

or by Lemma 2.1,

$$\left\{\frac{n-2}{2}(a-a')+2(b-b')\right\}\int_M G_{ji}\rho^j\rho^i dV$$
$$=\frac{1}{4}\{4a-(n+6)a'\}\int_M \rho^2 f dV+\frac{1}{4}\{4b-(n+6)b'\}\int_M \rho^2 g dV \tag{4.19}$$
$$-\frac{1}{4n}\int_M \mathscr{L}\mathscr{L}\left(af+\frac{n-1}{K}a'\Delta f+bg+\frac{n-1}{K}b'\Delta g\right)dV$$
$$+\frac{(n-1)(n+2)}{4K}\int_M \rho_i\rho^i(a'f+b'g)dV\,,$$

a, a', b, b' being constants. Now we choose these constants so as to have

$$\alpha_0=a\,,\quad \alpha_1=\frac{n-1}{K}a'\,,\quad \beta_0=b\,,\quad \beta_1=\frac{n-1}{K}b'\,. \tag{4.20}$$

Then from (4.16) we find

$$4a-(n+6)a'\geq 0\,,\qquad a'\geq 0\,,$$
$$4b-(n+6)b'\geq 0\,,\qquad b'\geq 0\,,$$

and

$$4(a-a')\geq(n+2)a'\geq 0\,,$$
$$4(b-b')\geq(n+2)b'\geq 0\,,$$

and consequently

$$\frac{n-2}{2}(a-a')+2(b-b')\geq 0\,,$$

the equality sign occurring when and only when $a=a'=b=b'=0$, that is, $\alpha_0=\alpha_1=\beta_0=\beta_1=0$.

Thus from the assumption and (4.19), we have

$$\int_M G_{ji}\rho^j\rho^i dV \geq 0 ,$$

and consequently, by theorem D, M is isometric to a sphere.

Remark 4.1. If

$$\mathscr{L}(\alpha_0 f + \alpha_1 \Delta f + \beta_0 g + \beta_1 \Delta g) = \text{constant},$$

then (4.15) is automatically satisfied. But if $\mathscr{L}h = \text{constant}$ for a scalar field h in a compact space, the constant must be zero, because h attains an extreme value at a certain point of the space at which $\mathscr{L}h = v^i \nabla_i h = 0$. The same remark applies to Theorems E, 4.1, 5.1, 6.1, 6.2 and 6.4.

5. A theorem similar to that of Hsiung

To obtain Theorem F, Hsiung [2] used the tensor

$$aZ_{kjih} + bg_{kh}G_{ji} ,$$

but we would like to use here the tensor

$$W_{kjih} = aZ_{kjih} + \frac{b}{n-2}(g_{kh}G_{ji} - g_{jh}G_{ki} + G_{kh}g_{ji} - G_{jh}g_{ki}) , \tag{5.1}$$

a and b being constants.

It is easily seen that

$$W_{kjih}g^{kh} = (a+b)G_{ji} , \tag{5.2}$$

and that, when $a + b = 0$,

$$W_{kjih} = aC_{kjih} , \tag{5.3}$$

where

$$\begin{aligned} C_{kjih} = K_{kjih} &- \frac{1}{n-2}(g_{kh}K_{ji} - g_{jh}K_{ki} + K_{kh}g_{ji} - K_{jh}g_{ki}) \\ &+ \frac{K}{(n-1)(n-2)}(g_{kh}g_{ji} - g_{jh}g_{ki}) \end{aligned} \tag{5.4}$$

is the covariant Weyl conformal curvature tensor.

In general, we have

$$W_{kjih}W^{kjih} = a^2 Z_{kjih}Z^{kjih} + \frac{4(2a+b)b}{n-2} G_{ji}G^{ji} , \tag{5.5}$$

and for the case $a + b = 0$ we have

$$W_{kjih}W^{kjih} = a^2 C_{kjih}C^{kjih} . \tag{5.6}$$

Using the tensor W_{kjih} defined above we can obtain

Theorem 5.1. *Suppose that a compact orientable M of dimension $n > 2$ with $K =$ const. admits an infinitesimal nonhomothetic conformal transformation v^h. If*

$$\mathscr{L}\mathscr{L}(W_{kjih}W^{kjih}) \leq 0 , \tag{5.7}$$

or equivalently,

$$(n-2)a^2\mathscr{L}\mathscr{L}(Z_{kjih}Z^{kjih}) + 4(2a+b)b\mathscr{L}\mathscr{L}(G_{ji}G^{ji}) \leq 0 , \tag{5.8}$$

a and b being constants such that $a + b \neq 0$, M is isometric to a sphere.

To prove this theorem, we need the following

Lemma 5.1. *For an infinitesimal conformal transformation v^h in M: $\mathscr{L}g_{ji} = 2\rho g_{ji}$, we have*

$$\begin{aligned} \mathscr{L}W_{kjih} = {} & 2a\rho Z_{kjih} + \frac{2b\rho}{n-2}(g_{kh}G_{ji} - g_{jh}G_{ki} + G_{kh}g_{ji} - G_{jh}g_{ki}) \\ & - (a+b)(g_{kh}\nabla_j\rho_i - g_{jh}\nabla_k\rho_i + \nabla_k\rho_h g_{ji} - \nabla_j\rho_h g_{ki}) \\ & + \frac{2(a+b)}{n}\Delta\rho(g_{kh}g_{ji} - g_{jh}g_{ki}) . \end{aligned} \tag{5.9}$$

Proof. This follows from (1.10), (1.11) and

$$\mathscr{L}Z_{kjih} = \mathscr{L}(Z_{kji}{}^t g_{th}) = (\mathscr{L}Z_{kji}{}^t)g_{th} + 2\rho Z_{kjih} . \tag{5.10}$$

Lemma 5.2. *For an infinitesimal conformal transformation v^h in M: $\mathscr{L}g_{ji} = 2\rho g_{ji}$, we have*

$$(\mathscr{L}W_{kjih})w^{kjih} = 2\rho W_{kjih}W^{kjih} - 4(a+b)^2 G_{ji}\nabla^j\rho^i . \tag{5.11}$$

Proof. This follows from (5.5) and (5.9).

Lemma 5.3. *For an infinitesimal conformal transformation v^h in M, we have*

$$\mathscr{L}(W_{kjih}W^{kjih}) = -4\rho W_{kjih}W^{kjih} - 8(a+b)^2 G_{ji}\nabla^j\rho^i . \tag{5.12}$$

Proof. This follows from (5.11) and

$$\mathscr{L}(W_{kjih}W^{kjih}) = 2(\mathscr{L}W_{kjih})W^{kjih} - 8\rho W_{kjih}W^{kjih} .$$

Lemma 5.4. *For an infinitesimal conformal transformation v^h in M with $K =$ constant, we have*

$$
\begin{aligned}
&8(a+b)^2\nabla^j(G_{ji}\rho\rho^i) \\
&\quad = 8(a+b)^2G_{ji}\rho^j\rho^i - 4\rho^2W_{kjih}W^{kjih} - \rho\mathscr{L}(W_{kjih}W^{kjih}) .
\end{aligned}
\tag{5.13}
$$

Proof. This follows from $\nabla^jG_{ji} = 0$ and (5.12).

Lemma 5.5. *If a compact orientable M of dimension $n > 2$ with $K = $ const. admits an infinitesimal nonhomothetic conformal transformation v^h, then we have*

$$
\begin{aligned}
&8(a+b)^2\int_M G_{ji}\rho^j\rho^i dV \\
&\quad = 4\int_M \rho^2W_{kjih}W^{kjih}dV + \int_M \rho\mathscr{L}(W_{kjih}W^{kjih})dV \\
&\quad = 4\int_M \rho^2W_{kjih}W^{kjih}dV - \frac{1}{n}\int_M \mathscr{L}\mathscr{L}(W_{kjih}W^{kjih})dV .
\end{aligned}
\tag{5.14}
$$

Proof. This follows from (5.13) by integrating both sides over M and using Lemma 2.1.

Proof of the theorem. If $\mathscr{L}\mathscr{L}(W_{kjih}W^{kjih}) \leq 0$, and $a + b \neq 0$, then from (5.14) we have

$$\int_M G_{ji}\rho^j\rho^i dV \geq 0 .$$

Thus by Theorem D, M is isometric to a sphere.

6. Characterizations of conformally flat spaces

Theorem 6.1. *If a compact orientable M of dimension $n > 3$ admits an infinitesimal conformal transformation v^h: $\mathscr{L}g_{ji} = 2\rho g_{ji}$ such that ρ does not vanish on any n-dimensional domain and*

$$\mathscr{L}\mathscr{L}h < 0, \qquad h = C_{kjih}C^{kjih} , \tag{6.1}$$

then M is conformally flat.

Proof. Multiplying (5.12), with $a + b = 0$, by ρ and integrating the resulting equation over M, we find

$$0 = 4\int_M \rho^2 h dV + \int_M \rho\mathscr{L}h dV ,$$

or by Lemma 2.1,

$$(6.2) \qquad 0 = 4\int_M \rho^2 h dV - \frac{1}{n}\int_M \mathcal{L}\mathcal{L} h dV .$$

(6.2) implies

$$\int_M \rho^2 h dV \leq 0 ,$$

from which $\rho^2 h = 0$, or by the assumption of the theorem, $h = 0$, that is, $C_{kjih} = 0$, which shows that M is conformally flat.

Remark 6.1. If $\mathcal{L}h =$ constant in a compact space, we have $\mathcal{L}h = 0$. On the other hand, if $\mathcal{L}h = 0$ in a general Riemannian space, from $\mathcal{L}h + 4\rho h = 0$ we find $h = 0$, which shows that the space is conformally flat.

Theorem 6.2. *Under the same assumptions as in Theorem* 6.1, *if* $K =$ *const. and* (6.1) *is replaced by*

$$(6.3) \qquad \mathcal{L}\left\{\sum_{k=0}^{l} \alpha_k \left(-\frac{n-1}{K}\right)^k \Delta^k(\mathcal{L}h)\right\} \leq 0 ,$$

l being a nonnegative integer and α_k constants such that $\sum_{k=0}^{l} \alpha_k > 0$, *then M is conformally flat.*

Proof. Similarly, as in the proof of Theorem 4.1 we can obtain

$$\begin{aligned} 0 = 4\int_M (\alpha_0 + \alpha_1 + \cdots + \alpha_l)\rho^2 h dV \\ + \int_M \rho\left\{\alpha_0 \mathcal{L}h + \alpha_1\left(-\frac{n-1}{K}\right)\Delta(\mathcal{L}h) + \cdots \right. \\ \left. + \alpha_l\left(-\frac{n-1}{K}\right)^l \Delta^l(\mathcal{L}h)\right\} dV , \end{aligned}$$

or, by virtue of Lemma 2.1,

$$(6.4) \qquad \begin{aligned} 0 = 4\int_M (\alpha_0 + \alpha_1 + \cdots + \alpha_l)\rho^2 dV \\ - \frac{1}{n}\int_M \mathcal{L}\left\{\alpha_0 \mathcal{L}h + \alpha_1\left(-\frac{n-1}{K}\right)\Delta(\mathcal{L}h) + \cdots \right. \\ \left. + \alpha_l\left(-\frac{n-1}{K}\right)^l \Delta^l(\mathcal{L}h)\right\} dV , \end{aligned}$$

$\alpha_0, \alpha_1, \cdots, \alpha_l$ being constants such that $\sum_{k=0}^{l} \alpha_k > 0$. Thus by (6.3), we have $\int_M \rho^2 h dV = 0$, from which $h = 0$ and consequently $C_{kjih} = 0$.

Theorem 6.3. *Under the same assumptions as in Theorem* 6.1, *if* $K =$ *const. and* (6.1) *is replaced by*

$$\alpha_0 h - \alpha_1 \Delta h = c \text{ (constant)}, \tag{6.5}$$

α_0 *and* α_1 *being positive constants such that*

$$\frac{8K}{n-1}\alpha_1 > (n-6)\alpha_0 \geq 0\,, \qquad \text{for } n > 6\,, \tag{6.6}$$

then M is conformally flat.

Proof. Similarly, as in the proof of Theorem 4.2 we can obtain

$$0 = \int_M \rho^2[\{8a' - (n-6)a\}h + (n+2)c]dV\,. \tag{6.7}$$

Now, the constant $8a' - (n-6)a$ is positive for $n \leq 6$. Since

$$8a' - (n-6)a = \frac{8K}{n-1}\alpha_1 - (n-6)\alpha_0\,,$$

by (6.6) this constant is also positive for $n > 6$.

On the other hand, from (6.5) we have

$$\alpha_0 \int_M h dV = c \int_M dV\,,$$

which shows that c is a nonnegative constant.

Thus from (6.7) we see that $h = 0$ and consequently $C_{kjih} = 0$.

Theorem 6.4. *Under the same assumption as in Theorem* 6.1, *if* $K =$ *const. and* (6.1) *is replaced by*

$$\mathcal{L}\mathcal{L}(\alpha_0 h + \alpha\, \Delta h) \leq 0\,, \tag{6.8}$$

α_0 *and* α_1 *being constants such that*

$$\frac{4(n-1)}{K}\alpha_0 > (n+6)\alpha_1 \geq 0\,, \tag{6.9}$$

then M is conformally flat.

Proof. Similarly, as in the proof of Theorem 4.3 we can obtain

$$\begin{aligned} 0 = \{4a - (n+6)a'\}\int_M \rho^2 h dV - \frac{1}{n}\int_M \mathcal{L}\mathcal{L}\left(ah + \frac{n-1}{K}a'\Delta h\right)dV \\ + \frac{(n-1)(n+2)}{K}\int_M \rho_i\rho^i(a'h)dV\,, \end{aligned} \tag{6.10}$$

a and a' being constants. Now we choose these constants such that

(6.11) $$\alpha_0 = a\,, \qquad \alpha_1 = \frac{n-1}{K}a'\,.$$

Then from (6.9) we have

$$4a - (n+6)a' = 4\alpha_0 - (n+6)\frac{K}{n-1}\alpha_1 > 0\,.$$

We also have

$$\mathscr{L}\left(ah + \frac{n-1}{K}a'\Delta h\right) = \mathscr{L}(\alpha_0 h + \alpha_1 \Delta h) = \text{constant}\,.$$

Thus, from (6.10), we have $h = 0$ and consequently $C_{kjih} = 0$.

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